New Concept of First Quantization

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We derive a Lagrangian density of Dirac field by employing the local gauge invariance and the Maxwell equation as the fundamental principle. The only assumption made here is that the fermion field should have four components. The present derivation of the Dirac equation does not involve the first quantization, and therefore this study may present an alternative way of understanding the quantization procedure.

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I. INTRODUCTION

Dirac derived the Dirac equation by factorizing Einstein's dispersion relation such that the field equation becomes the first order in time derivative [1]. Namely, he factorized the relativistic dispersion relation employing four by four matrices

$$E^2 - \boldsymbol{p}^2 - m^2 = (E - \boldsymbol{p}\cdot\boldsymbol{\alpha} - m\beta)(E + \boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta) \quad (1.1)$$

where $\boldsymbol{\alpha}$ and β are Dirac matrices which satisfy the following anti-commutation relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \qquad \{\alpha_i, \beta\} = 0, \qquad \beta^2 = 1.$$

Here, i and j run $i, j = x, y, z$. In this case, Dirac equation becomes

$$\left(i\hbar\frac{\partial}{\partial t} + i\hbar\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} - m\beta\right)\psi = 0 \qquad (1.2)$$

where ψ denotes the wave function and should have four components since $\boldsymbol{\alpha}$ and β are four by four matrices,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \qquad (1.3)$$

In eq.(1.2), the first quantization condition

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \boldsymbol{p} \to -i\hbar\boldsymbol{\nabla} \qquad (1.4)$$

is employed. Even though the first quantization is not understood very well from the fundamental principle, the Dirac equation and the first quantization are, of course, consistent with experiment. Nevertheless, it should be interesting if one can derive the Dirac equation without involving the first quantization condition.

In this Letter, we derive the Lagrangian density of the fermion fields interacting with the gauge field A^μ. This is done by employing the Maxwell equation as the most fundamental equation and by requiring the local gauge invariance of the Lagrangian density, but the first quantization condition of eq.(1.4) is not employed at all. The basic assumption we have made, in addition to the local gauge invariance, is that the fermion field should have four components and the Lagrangian density should be written by Lorentz scalars.

In the present picture, we can immediately obtain two interesting results. The first consequence from the new picture is that the Klein-Gordon equation has lost its foundation. This can be easily seen, since the first quantization condition is not the fundamental principle any more, it cannot be used for the replacement of the energy and momentum by the differential operators in the squared relativistic dispersion relation. The second result is connected to the well-known wrong Hamiltonian which is obtained by the canonical quantization procedure in polar coordinates. This is now understood well since the first quantization is obtained from the Dirac equation as the consequence by the identification of the momenta in terms of the differential operators.

II. LAGRANGIAN DENSITY FOR MAXWELL EQUATION

Now, we discuss a new way of deriving the Lagrangian density of Dirac field by employing the local gauge invariance and the Maxwell equation as the starting point. Since the Maxwell equation is a field equation, the first quantization is already done there. Therefore, we consider that the Maxwell equation should give a guide to constructing the Lagrangian density for fermion fields.

First, we start from the Lagrangian density that reproduces the Maxwell equation

$$\mathcal{L} = -gj_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (2.1)$$

where A^μ is the gauge field, and $F_{\mu\nu}$ is the field strength which is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

j_μ denotes the current density of matter field which couples with the electromagentic field. From the Lagrange equation, one obtains

$$\partial_\mu F^{\mu\nu} = g j^\nu \qquad (2.2)$$

which is just the Maxwell equation. For the case of no external current ($j^\mu = 0$), we obtain the equation for the gauge field $\boldsymbol{A}$

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)\boldsymbol{A} = 0 \qquad (2.3)$$

where we have made use of the Coulomb gauge fixing

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0.$$

It is clear that eq.(2.3) is a quantized equation for the gauge field, and therefore as stressed before, the Maxwell equation knows in advance the first quantization procedure.

III.   FOUR COMPONENT SPINOR

Now, we derive the kinetic energy term of the fermion Lagrangian density. First, we assume that the Dirac fermion field should be written by four component spinor of eq.(1.3). This is based on the observation that electron has spin degree of freedom which is two. In addition, there must be positive and negative energy states since it is a relativistic field, and therefore the fermion field should have 4 components. This is the only ansatz which is set by hand in the derivation of the Dirac equation.

A.   Matrix Elements

The matrix elements $\psi^\dagger \hat{O} \psi$ can be classified into 16 independent Lorentz invariant components as [2]

$$\bar{\psi}\psi : \text{scalar}, \qquad \bar{\psi}\gamma_5\psi : \text{pseudo} - \text{scalar} \qquad (3.1a)$$

$$\bar{\psi}\gamma_\mu\psi : 4 \text{ component vector} \qquad (3.1b)$$

$$\bar{\psi}\gamma_\mu\gamma_5\psi : 4 \text{ component axial} - \text{vector} \qquad (3.1c)$$

$$\bar{\psi}\sigma_{\mu\nu}\psi, : 6 \text{ component tensor} \qquad (3.1d)$$

where $\bar{\psi}$ is defined as $\bar{\psi} = \psi^\dagger \gamma_0$. These properties are determined by mathematics, and therefore the vector current representation has nothing to do with physics.

B.   Shape of Vector Current

From the invariance consideration, one finds that the vector current j_μ must be written as

$$j_\mu = C_0 \bar{\psi}\gamma_\mu\psi \qquad (3.2)$$

where C_0 is an arbitrary constant. Since we can renormalize the constant C_0 into the coupling constant g, we can set

$$C_0 = 1. \qquad (3.3)$$

IV.   SHAPE OF LAGRANGIAN DENSITY

Now, we make use of the local gauge invariance of the Lagrangian density, and we assume the following shape that may keep the local gauge invariance

$$\mathcal{L} = C_1 \bar{\psi}\partial_\mu\gamma^\mu\psi - g\bar{\psi}\gamma_\mu\psi A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (4.1)$$

where C_1 is a constant.

Now, we require that the Lagrangian density should be invariant under the local gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\chi \qquad (4.2a)$$

$$\psi \to e^{-ig\chi}\psi. \qquad (4.2b)$$

In this case, it is easy to find that the constant C_1 must be

$$C_1 = i. \qquad (4.3)$$

Here, the constant $\hbar$ should be included implicitly into the constant C_1. The determination of $\hbar$ can be done only when one compares calculated results with experiment such as the spectrum of hydrogen atom.

The Lagrangian density of eq.(4.1) still lacks the mass term. Since the mass term must be a Lorentz scalar, it should be described as

$$C_2\bar{\psi}\psi \qquad (4.4)$$

which is gauge invariant as well. This constant C_2 should be determined again by comparing the equation with experiment. By denoting C_2 as $(-m)$, we arrive at the Lagrangian density of a relativistic fermion which couples with the electromagnetic fields A_μ

$$\mathcal{L} = i\bar{\psi}\partial_\mu\gamma^\mu\psi - g\bar{\psi}\gamma_\mu\psi A^\mu - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (4.5)$$

This is just the Lagrangian density for Dirac field and gives the Dirac equation.

It is important to note that, in the procedure of deriving the Lagrangian density of eq.(4.5), we have not made use of the quantization condition of eq.(1.4). Instead, the first quantization is automatically done by the local gauge condition since the Maxwell equation knows the first quantization in advance.

V.   TWO COMPONENT SPINOR

The present derivation of the Dirac equation shows that the current density that can couple to the gauge field A^μ must be rather limited. Here, we discuss a possibility of finding field equation for the two component spinor [3].

When the field has only two components,

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad (5.1)$$

then one can prove that one cannot make the current j_μ that couples with the gauge field A_μ. This can be easily seen since the matrix elements $\phi^\dagger \hat{O} \phi$ can be classified into 4 independent variables as

$$\phi^\dagger \phi : \text{scalar}, \qquad \phi^\dagger \sigma_k \phi : \quad 3 \text{ component vector}. \quad (5.2)$$

Therefore, there is no chance to make four vector currents which may couple to the gauge field A_μ.

This way of making the Lagrangian density indicates that it should be difficult to find a Lagrangian density of relativistic bosons which should have two components.

VI.   SOME RESULTS FROM THE NEW PICTURE

In the present derivation of the Dirac equation, there is no need of the first quantization. Instead, the gauge principle is taken to be the most important guiding principle. At least, the first quantization does not have to be taken as the fundamental principle from which one can derive quantum mechanical equations. Here, one can derive the correspondence between $(E, \boldsymbol{p})$ and the differential operators $(i\hbar \frac{\partial}{\partial t}, -i\hbar \boldsymbol{\nabla})$ as a consequence from the Dirac equation

$$\text{Dirac equation} \Rightarrow \quad E \to i\hbar \frac{\partial}{\partial t}, \quad \boldsymbol{p} \to -i\hbar \boldsymbol{\nabla}.$$

However, this replacement is not a fundamental principle any more as we saw above.

Now, we may find some results from the new picture of quantum mechanics. Below we discuss some simple but interesting predictions from the new concept.

A.   No Klein-Gordon Equation

The procedure of obtaining the Schrödinger equation in terms of the replacement of the momentum and energy by the differential operators in the classical Hamiltonian can be justified since the Dirac equation can be reduced to the Schrödinger equation in the non-relativistic limit. However, one cannot apply the relation eq.(1.4) to the energy dispersion relation with relativistic kinematics since eq.(1.4) is not the fundamental principle any more.

Historically, the Klein-Gordon equation was obtained by making use of the squared relativistic dispersion relation

$$E^2 = \boldsymbol{p}^2 + m^2 \qquad (6.1)$$

and one replaced the energy and momentum by eq.(1.4). This leads to the following equation

$$E^2 = \boldsymbol{p}^2 + m^2 \Rightarrow -\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left(-\hbar^2 \boldsymbol{\nabla}^2 + m^2 \right) \psi \quad (6.2)$$

which is just the Klein-Gordon equation. However, this is only justified when the replacement of eq.(1.4) is the fundamental principle. If eq.(1.4) is only derived from the Dirac equation as a consequence of the field equation, then the Klein-Gordon equation cannot be justified any more. In other words, there is no way to derive the Klein-Gordon equation from any where, or at least, the Klein-Gordon equation has lost its foundation since the first quantization condition is not the fundamental principle. This is quite reasonable since there is no physical meaning of eq.(6.1) and the Klein-Gordon equation is just derived in analogy to the derivation of the Schrödinger equation.

In addition, in spite of the fact that the Klein-Gordon equation is obtained by analogy to the non-relativistic dispersion relation, the Klein-Gordon field does not have any corresponding field in the non-relativistic limit. This peculiar behavior of the Klein-Gordon field is known, but it has never been discussed seriously. In this respect, the non-existence of the Klein-Gordon field must be quite natural even though bosons should, of course, exist as composite objects of fermion and anti-fermion bound states [4].

B.   Incorrect Quantization in Polar Coordinates

In quantum mechanics, one learns that one should not quantize the classical Hamiltonian in polar coordinates. In order to see it more explicitly, one writes a free particle Hamiltonian in classical mechanics with its mass m

$$H = \frac{p_r{}^2}{2m} + \frac{p_\theta{}^2}{2mr^2} + \frac{p_\varphi{}^2}{2mr^2 \sin^2 \theta} \qquad (6.3)$$

where p_r, p_θ, p_φ are the generalized momenta in polar coordinates. In this case, if one quantizes the free particle Hamiltonian in classical mechanics with the quantization conditions

$$[r, p_r] = i\hbar, \quad [\theta, p_\theta] = i\hbar, \quad [\varphi, p_\varphi] = i\hbar \qquad (6.4)$$

then one obtains

$$H = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right). \qquad (6.5)$$

As is well known, this is not a correct Hamiltonian for a free particle in the polar coordinates, and the correct

Hamiltonian for a free particle is, of course, given as

$$H = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right)$$

$$-\frac{\hbar^2}{2mr^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \qquad (6.6)$$

which is obtained by transforming the Schrödinger equation in Cartesian coordinate into the polar coordinate. However, one does not understand any reasons why one cannot quantize the Hamiltonian in the polar coordinates as long as one starts from the canonical formalism with the canonical quantization. In quantum mechanics textbooks, one learns as an empirical fact that the transformation of the Cartesian to polar coordinates should be done after the quantization in the Cartesian coordinates is made in the Hamiltonian.

Now, one can understand the reason why the quantization condition is valid for the Cartesian coordinates, but not for the polar coordinates. That is, in the present picture, the differential operators first appear in the Dirac equation and then one identifies the momentum as the corresponding differential operator. Therefore, one can obtain the Schrödinger equation from the Dirac equation, but not from the Hamiltonian of the classical mechanics by replacing the generalized momentum by the corresponding differential operator. In this sense, the canonical formalism of the classical mechanics is mathematically interesting, but it is not useful for quantum mechanics.

VII. DISCUSSIONS

The quantization condition of $[x_i, p_j] = i\hbar\delta_{ij}$ is always the starting point of quantum mechanics. One learns in the textbooks that this condition is probably more fundamental than the momentum operator replacement of $p_k \to -i\hbar\frac{\partial}{\partial x_k}$. However, one cannot explain why the quantization condition is postulated, apart from the fact that the Schrödinger equation can be obtained from the Hamiltonian of classical mechanics.

In this Letter, we have presented a new interpretation of the first quantization. Instead of the normal quantization, we have shown that the Lagrangian density of the Dirac field can be obtained from the local gauge invariance and the Maxwell equation. The first quantization procedure can be obtained from the Dirac equation as the consequence of the identification of the energy and momentum by the differential operators.

In the historical derivation of the Schrödinger equation, one started from the Newton equation or the classical Hamiltonian dynamics. In this case, one had to assume the quantization condition of $\boldsymbol{p} \to -i\hbar\boldsymbol{\nabla}$. Here, we claim that the Dirac equation can be first obtained, and from the Dirac equation, one can derive the Schrödinger equation by the Foldy-Wouthuysen-Tani transformation in the non-relativistic limit. From the Schrödinger equation, one can derive the Newton equation when one takes the expectation value, which is the Ehrenfest theorem.

The present work is essentially based on the belief that the Maxwell equation and the local gauge invariance are the most fundamental principle. Once this standpoint is accepted, it is straightforward to derive the Dirac equation without the first quantization ansatz. Therefore, this procedure of first deriving the Dirac equation before the Schrödinger equation may present a new conceptual development in quantum mechanics, and one may learn novel aspects in quantum mechanics in a future study.

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